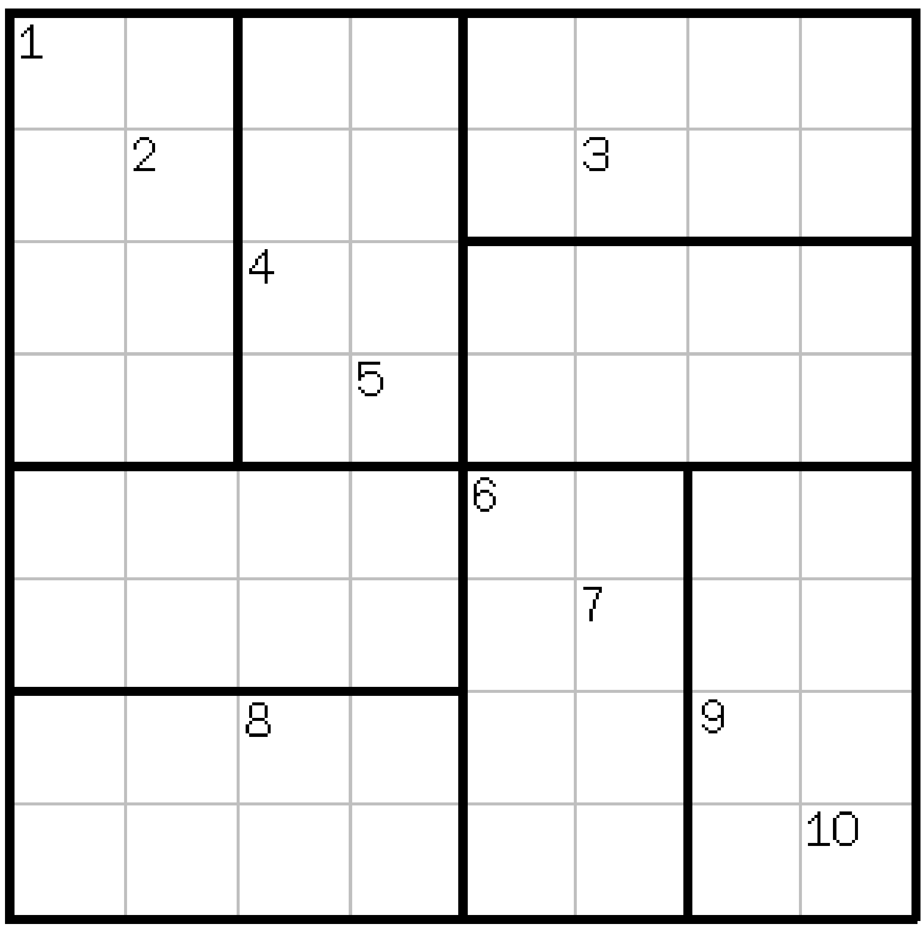
6th Annual Lexington Math Tournament Team Round

March 28, 2014

Name ______________________________ Team ______________________________



This page is intentionally left blank.

1 Potpourri[85]

The answers to each of the ten questions in this section are integers containing only the digits 1 through 8, inclusive. Each answer can be written into the grid on the answer sheet, starting from the cell with the problem number, and continuing across or down until the entire answer has been written. Answers may cross dark lines. If the answers are correctly filled in, it will be uniquely possible to write an integer from 1 to 8 in every cell of the grid, so that each number will appear exactly once in every row, every column, and every marked 2 by 4 box. You will get 7 points for every correctly filled answer, and a 15 point bonus for filling in every gridcell. It will help to work back and forth between the grid and the problems, although every problem is uniquely solvable on its own.

Please write clearly within the boxes. No points will be given for a cell without a number, with multiple numbers, or with illegible handwriting.

1 ACROSS: Jack puts 10 red marbles, 8 green marbles and 4 blue marbles in a bag. If he takes out 11 marbles, what is the expected number of green marbles taken out?

2 DOWN: What is the closest integer to $6\sqrt{35}$?

3 ACROSS: Alan writes the numbers 1 to 64 in binary on a piece of paper without leading zeroes. How many more times will he have written the digit 1 than the digit 0?

4 ACROSS: Integers a and b are chosen such that $-50 < a, b \leq 50$. How many ordered pairs (a, b) satisfy the below equation?

$$(a + b + 2)(a + 2b + 1) = b$$

5 DOWN: Zach writes the numbers 1 through 64 in binary on a piece of paper without leading zeroes. How many times will he have written the two-digit sequence "10"?

6 ACROSS: If you are in a car that travels at 60 miles per hour, \$1 is worth 121 yen, there are 8 pints in a gallon, your car gets 10 miles per gallon, a cup of coffee is worth \$2, there are 2 cups in a pint, a gallon of gas costs \$1.50, 1 mile is about 1.6 kilometers, and you are going to a coffee shop 32 kilometers away for a gallon of coffee, how much, in yen, will it cost?

7 DOWN: Clive randomly orders the letters of "MIXING THE LETTERS, MAN". If $\frac{p}{m^n q}$ is the probability that he gets "LMT IS AN EXTREME THING" where p and q are odd integers, and m is a prime number, then what is $m + n$?

8 ACROSS: Joe is playing darts. A dartboard has scores of 10, 7, and 4 on it. If Joe can throw 12 darts, how many possible scores can he end up with?

9 ACROSS: What is the maximum number of bounded regions that 6 overlapping ellipses can cut the plane into?

10 DOWN: Let ABC be an equilateral triangle, such that A and B both lie on a unit circle with center O. What is the maximum distance between O and C? Write your answer be in the form $\frac{a\sqrt{b}}{c}$ where b is not divisible by the square of any prime, and a and c share no common factor. What is abc?

2 Long Answer[115]

This section deals with colleges and campus maps. The problems will use some of the following definitions:

A *college* is collection of school buildings, paths joining some pairs of them.

A *path* must be a **straight line** connecting two distinct buildings which are not connected by any other path, and **no path may intersect any other path**.

It is called a *campus* if it is possible to walk on paths from any building to any other building. Assume that school buildings are infinitesimally small points.

1 [7] What is the most amount of paths in a college with 4 buildings?

2 [9] Prove that removing a path from a campus will either cut it into two or one campuses and give an example of both.

3 [11] Prove that you can get from any building to any other building on a campus by traveling through less than n paths.

4 [13] With proof, what is the most amount of paths in a college with 2015 buildings?

Zach is reading through some college brochures. Some of the advertised features look too good to be true. For each of the following campuses, show an example of it, or prove that it cannot possibly exist. For problems 7 and 9, if you think that such a campus exists, you do not need to draw it. Instead, provide a way to construct it, and prove that it meets the requirements.

5 [11] A campus with at least two buildings, that doesn't have any pair of buildings that have the same amount of paths leading out of them.

6 [13] A campus such that every building has at least 5 paths leading from it.

7 [15] A campus such that every building has at least 2015 paths leading from it.

8 [17] A campus with math buildings and humanities buildings, such that every math building has paths leading to at least 5 humanities buildings, and every humanities building has paths leading to at least 3 math buildings.

9 [19] An infinite campus with at least one path that can have one edge removed to become two campuses that are entirely identical to the first campus. To be considered identical, two campuses do not need to have the same size or shape; rather, it must be possible to map buildings from one campus to the other such that every two buildings that have a path between them in the first campus have a path between them in the second.